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The first eigenvalue of the transversal Dirac operator☆

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Abstract

On a foliated Riemannian manifold with a transverse spin structure, we give a lower bound for the square of the eigenvalues of the transversal Dirac operator. We prove, in the limiting case, that the foliation is a minimal, transversally Einsteinian with constant transversal scalar curvature.

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1. Introduction

In 1963, Lichnerowicz [10] proved that on a Riemannian spin manifold the square of the Dirac operator D is given by

$$D^2 = \Delta + \tfrac{1}{4}\sigma,$$

where Δ is the positive spinor Laplacian and σ the scalar curvature. In 1980, Friedrich [4] gave a lower bound for the square of the eigenvalues of the Dirac operator in the above equation, as follows:

$$\lambda^2 \geq \frac{n}{4(n-1)}\sigma_0,$$

where $\sigma_0 = \min \sigma$. He also proved, in the limiting case, that the manifold is an Einstein.

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In 1988, Brüning and Kamber [2] defined the transversal Dirac operator $D_{\rm tr}$ on M and proved the following equation:

$$D_{\rm tr}^2 = \nabla_{\rm tr}^* \nabla_{\rm tr} + \mathcal{R}_\nabla + \mathcal{K}_\nabla,$$

where $\mathcal{R}_\nabla$ is an endomorphism containing the curvature data and K_∇ a function containing the mean curvature of the leaves.

In this paper, we study the transversal Dirac operator $D_{\rm tr}$ and its eigenvalue on the foliated Riemannian manifold M.

This paper is organized as follows. In Section 2, we review the known facts on the foliated Riemannian manifold. In Section 3, we study some basic properties of the transversal Dirac operator $D_{\rm tr}$. In Section 4, we give a lower bound for the square of the eigenvalues of the transversal Dirac operator $D_{\rm tr}$. In Section 5, we prove, in the limiting case, that the foliation is a minimal, transversally Einsteinian with constant transversal scalar curvature. The technique we use is similar to the one in [4] if we do not consider the mean curvature of the foliation.

2. Preliminaries and known facts

Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$.

We recall the exact sequence

$$0 \to L \to TM \stackrel{\pi}{\to} Q \to 0$$

determined by the tangent bundle L and the normal bundle Q of $\mathcal{F}$. The assumption of g_M to be a bundle-like metric means that the induced metric g_Q on the normal bundle $Q \cong L^\perp$ satisfies the holonomy invariance condition $\theta(X) g_Q = 0$ for all $X \in \Gamma L$, where $\theta(X)$ denotes the Lie derivative with respect to X.

For a distinguished chart $\mathcal{U} \subset M$ the leaves of $\mathcal{F}$ in $\mathcal{U}$ are given as the fibers of a Riemannian submersion $f : \mathcal{U} \to \mathcal{V} \subset N$ onto an open subset $\mathcal{V}$ of a model Riemannian manifold N.

For overlapping charts $U_\alpha \cap U_\beta$, the corresponding local transition functions $\gamma_{\alpha\beta} = f_\alpha \circ f_\beta^{-1}$ on N are isometries. Further, we denote by ∇ the canonical connection of the normal bundle $Q = TM/L$ of $\mathcal{F}$. It is defined by

$$\nabla_X s = \pi([X, Y_s]) \quad \text{for } X \in \Gamma L, \qquad \nabla_X s = \pi(\nabla^M_X Y_s) \quad \text{for } X \in \Gamma L^\perp, \tag{2.1}$$

where $s \in \Gamma Q$, and $Y_s \in \Gamma L^\perp$ corresponding to s under the canonical isomorphism $L^\perp \cong Q$. The connection ∇ is metric and torsion-free. It corresponds to the Riemannian connection of the model space N [7]. The curvature R^∇ of ∇ is defined by

$$R^\nabla_{XY} = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \quad \text{for } X, Y \in TM.$$

Since $\mathrm{i}(X) R^\nabla = 0$ for any $X \in \Gamma L$ [7], we can define the (transversal) Ricci curvature

$\rho^\nabla : \Gamma Q \to \Gamma Q$ and the (transversal) scalar curvature σ^∇ of $\mathcal{F}$ by

$$\rho^\nabla(s) = \sum_a R^\nabla_{sE_a} E_a, \qquad \sigma^\nabla = \sum_a g_Q(\rho^\nabla(E_a), E_a),$$

where $\{E_a\}_{a=1,\dots,q}$ is an orthonormal basis for Q. $\mathcal{F}$ is said to be (transversally) *Einsteinian* if the model space N is Einsteinian, i.e.,

$$\rho^\nabla = \frac{1}{q}\sigma^\nabla \cdot \mathrm{id} \tag{2.2}$$

with constant transversal scalar curvature σ^∇.

The *second fundamental form* of α of $\mathcal{F}$ is given by

$$\alpha(X, Y) = \pi(\nabla^M_X Y) \quad \text{for } X, Y \in \Gamma L. \tag{2.3}$$

It is trivial that α is Q-valued, bilinear and symmetric.

The *mean curvature vector field* of $\mathcal{F}$ is then defined by

$$\tau = \sum_i \alpha(E_i, E_i), \tag{2.4}$$

where $\{E_i\}_{i=1,\dots,p}$ is an orthonormal basis of L. The dual form κ, the *mean curvature form* for L, is then given by

$$\kappa(X) = g_Q(\tau, X) \quad \text{for } X \in \Gamma Q. \tag{2.5}$$

The foliation $\mathcal{F}$ is said to be *minimal* (or *harmonic*) if $\kappa = 0$.

Let $\Omega^r_{\mathrm{B}}(\mathcal{F})$ be the space of all *basic r-forms*, i.e.,

$$\Omega^r_{\mathrm{B}}(\mathcal{F}) = \{\phi \in \Omega^r(M) | \mathrm{i}(X)\phi = 0, \theta(X)\phi = 0 \ \text{ for } X \in \Gamma L\}.$$

The foliation $\mathcal{F}$ is said to be *isoparametric* if $\kappa \in \Omega^1_{\mathrm{B}}(\mathcal{F})$. We already know that κ is closed, i.e., $\mathrm{d}\kappa = 0$ if $\mathcal{F}$ is isoparametric [11]. Since the exterior derivative preserves the basic forms (i.e., $\theta(X)\,\mathrm{d}\phi = 0$ and $\mathrm{i}(X)\,\mathrm{d}\phi = 0$ for $\phi \in \Omega^r_{\mathrm{B}}(\mathcal{F})$), the restriction $d_{\mathrm{B}} = d|_{\Omega^*_{\mathrm{B}}(\mathcal{F})}$ is well defined. The adjoint operator δ_{B} of d_{B} is given by

$$\delta_{\mathrm{B}}\phi = (-1)^{q(r+1)+1}\bar{*}(d_{\mathrm{B}} - \kappa_{\mathrm{B}}\wedge)\bar{*}\phi \quad \text{for } \phi \in \Omega^r_{\mathrm{B}}(\mathcal{F}), \tag{2.6}$$

where κ_{B} is the basic component of κ and $\bar{*} : \Omega^r_{\mathrm{B}}(\mathcal{F}) \to \Omega^{q-r}_{\mathrm{B}}(\mathcal{F})$ a star operator [1]. When κ is basic, this formula reduces to Theorem 12.10 in [11]. The *basic Laplacian* acting on $\Omega^*_{\mathrm{B}}(\mathcal{F})$ is defined by

$$\Delta_{\mathrm{B}} = d_{\mathrm{B}}\delta_{\mathrm{B}} + \delta_{\mathrm{B}} d_{\mathrm{B}}.$$

If $\mathcal{F}$ is the foliation by points of M, the basic Laplacian is the ordinary Laplacian.

3. The transversal Dirac operator

Let E be a complex Hermitian *foliated* bundle [8] over M which is a Clifford module over $\mathrm{Cl}(Q)$, the *transversal Clifford algebra* of $\mathcal{F}$. We assume that E carries a Hermitian metric $\langle\cdot,\cdot\rangle$ and an orthogonal connection ∇^E such that

1. The Clifford multiplication '$\cdot$' by unit vectors in Q is orthogonal, i.e., at each point $x\in M$,

$$\langle e\cdot\Phi,\Psi\rangle+\langle\Phi,e\cdot\Psi\rangle=0 \tag{3.1}$$

for all $\Phi,\Psi\in E_x$ and all unit vectors $e\in Q_x$.

2. The covariant derivative ∇^E on E is a module derivation, i.e.,

$$\nabla^E(s\cdot\Phi)=(\nabla s)\cdot\Phi+s\cdot(\nabla^E\Phi) \tag{3.2}$$

for all $s\in\Gamma\,\mathrm{Cl}(Q)$ and all $\Phi\in\Gamma E$. If it does not cause any confusion, we will henceforward use $\nabla=\nabla^E$. Taking $\hat\pi$ to denote the projection

$$\hat\pi : C^\infty(T^*M\otimes E)\to C^\infty(Q^*\otimes E)\cong C^\infty(Q\otimes E),$$

we define the *transversal Dirac operator* D'_{tr} by

$$D'_{\mathrm{tr}}=\cdot\circ\hat\pi\circ\nabla.$$

If $\{E_a\}_{a=1,\dots,q}$ is taken to be a local orthonormal basic frame in Q, then

$$D'_{\mathrm{tr}}=\sum_a E_a\cdot\nabla_{E_a}.$$

In [3], it was shown that the formal adjoint D'^*_{tr} is given by $D'^*_{\mathrm{tr}}=D'_{\mathrm{tr}}-\kappa\cdot$ and that therefore

$$D_{\mathrm{tr}}=D'_{\mathrm{tr}}-\tfrac{1}{2}\kappa\cdot \tag{3.3}$$

is a symmetric, transversally elliptic differential operator, with symbol $\sigma_{D_{\mathrm{tr}}}$ satisfying $\sigma_{D_{\mathrm{tr}}}(x,\xi)=\xi$ for $\xi\in Q^*_x$ and $\sigma_{D_{\mathrm{tr}}}(x,\xi)=0$ for $\xi\in L^*_x$. We define the subspace $\Gamma_{\mathrm{B}}(E)$ of *basic* or *holonomy invariant* sections of E by

$$\Gamma_{\mathrm{B}}(E)=\{\Phi\in\Gamma E|\nabla_X\Phi=0\quad\text{for } X\in\Gamma L\}. \tag{3.4}$$

If we consider the vector bundle $E=\wedge Q^*\otimes C$, then we have

$$\Gamma_{\mathrm{B}}(E)=\Omega^*_{\mathrm{B}}(\mathcal{F})\otimes C. \tag{3.5}$$

From (3.3), we see that D_{tr} leaves $\Gamma_{\mathrm{B}}(E)$ invariant if and only if the foliation $\mathcal{F}$ is isoparametric, i.e., $\kappa\in\Omega^1_{\mathrm{B}}(\mathcal{F})$. Let $D_{\mathrm{b}}=D_{\mathrm{tr}}|_{\Gamma_{\mathrm{B}}(E)}:\Gamma_{\mathrm{B}}(E)\to\Gamma_{\mathrm{B}}(E)$. This operator D_{b} is called the *basic Dirac operator* on (smooth) basic sections $\Gamma_{\mathrm{B}}(E)$. We now define $\nabla^*_{\mathrm{tr}}\nabla_{\mathrm{tr}}:\Gamma E\to\Gamma E$ as

$$\nabla^*_{\mathrm{tr}}\nabla_{\mathrm{tr}}\Phi=-\sum_a\nabla^2_{E_a,E_a}\Phi+\nabla_\kappa\Phi, \tag{3.6}$$

where $\nabla^2_{v,w}=\nabla_v\nabla_w-\nabla_{\nabla_v w}$ for any $v,w\in TM$.

Proposition 3.1. *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a foliation $\mathcal{F}$ and a bundle-like metric g_M with respect to $\mathcal{F}$. Then*

$$\langle\langle \nabla_{\rm tr}^* \nabla_{\rm tr} \Phi, \Psi \rangle\rangle = \langle\langle \nabla_{\rm tr} \Phi, \nabla_{\rm tr} \Psi \rangle\rangle$$

for all $\Phi, \Psi \in \Gamma E$, where $\langle\langle \Phi, \Psi \rangle\rangle = \int_M \langle \Phi, \Psi \rangle$ is the inner product on E.

Proof. Fix $x \in M$ and choose an orthonormal basic frame $\{E_a\}$ with the property that $(\nabla E_a)_x = 0$ for all a. Then we have at the point x that for any Φ, Ψ,

$$\begin{aligned} \langle \nabla_{\rm tr}^* \nabla_{\rm tr} \Phi, \Psi \rangle &= -\sum_a \langle \nabla_{E_a} \nabla_{E_a} \Phi, \Psi \rangle + \langle \nabla_\kappa \Phi, \Psi \rangle \\ &= -\sum_a E_a \langle \nabla_{E_a} \Phi, \Psi \rangle + \sum_a \langle \nabla_{E_a} \Phi, \nabla_{E_a} \Psi \rangle + \langle \nabla_\kappa \Phi, \Psi \rangle \\ &= -{\rm div}_\nabla (v) + \langle \nabla_{E_a} \Phi, \nabla_{E_a} \Psi \rangle + \langle \nabla_\kappa \Phi, \Psi \rangle, \end{aligned} \quad (3.7)$$

where $v \in \Gamma Q$ defined by the condition that $g_Q(v, w) = \langle \nabla_w \Phi, \Psi \rangle$ for all $w \in \Gamma Q$. The last line of (3.7) is proved as follows. At $x \in M$,

$${\rm div}_\nabla (v) = \sum_a g_Q(\nabla_{E_a} v, E_a) = \sum_a E_a g_Q(v, E_a) = \sum_a E_a \langle \nabla_{E_a} \Phi, \Psi \rangle.$$

By the Green's theorem on the foliated Riemannian manifold [12], we have

$$\int_M {\rm div}_\nabla (v) = \langle\langle \kappa, v \rangle\rangle = \langle\langle \nabla_\kappa \Phi, \Psi \rangle\rangle. \quad (3.8)$$

The result is due to integration of (3.7). □

We now define a canonical section $\mathcal{R}_\nabla$ of $\mathrm{Hom}(E, E)$ by the formula

$$\mathcal{R}_\nabla(\Phi) = \sum_{a<b} E_a \cdot E_b \cdot R^E_{E_a E_b} \Phi,$$

where R^E is the curvature tensor of E. If $\mathcal{F}$ is isoparametric, then we have the Bochner–Weitzenböck-type formula

$$D_{\rm tr}^2 \Phi = \nabla_{\rm tr}^* \nabla_{\rm tr} \Phi + \mathcal{R}_\nabla(\Phi) + \mathcal{K}_\nabla \Phi, \quad (3.9)$$

where $\mathcal{K}_\nabla = \frac{1}{2}\{-\delta\kappa + \frac{1}{2}|\kappa|^2\}$ [2,3,5]. On $\Gamma_{\rm B}(E)$, we have

$$D_{\rm b}^2 = \Delta|_{\Gamma_{\rm B}(E)}, \quad (3.10)$$

where $\Delta = \nabla^*\nabla + \mathcal{R}_\nabla + \mathcal{K}_\nabla$ is a strongly elliptic, symmetric operator of Laplace type. To prove theorems in this paper, it is useful to assume that κ is divergence-free, i.e., $\delta\kappa = 0$. Since κ is already closed, κ is a harmonic 1-form. We then have $\mathcal{K}_\nabla = \frac{1}{4}|\kappa|^2$ and the resulting local equation

$$\langle\langle D_{\rm b}^2 \Phi, \Phi \rangle\rangle = \|\nabla \Phi\|^2 + \langle\langle \mathcal{R}_\nabla(\Phi), \Phi \rangle\rangle + \tfrac{1}{4}\| |\kappa| \Phi \|^2 \quad (3.11)$$

implies transversal vanishing theorems for $\mathrm{Ker}(D_{\rm b})$ by the usual Bochner–Lichnerowicz argument, provided $\mathcal{R}_\nabla \geq 0$ and $\mathcal{R}_\nabla$ is positive at least at one point $x_0 \in M$ [2].

Lemma 3.2. *Let $\mathcal{F}$ be a Riemannian foliation. Then the operators d_{B} and δ_{B} on $\Omega^*_{\mathrm{B}}(\mathcal{F})$ are given by*

$$d_{\mathrm{B}} = \sum_a \theta_a \wedge \nabla_{E_a}, \qquad \delta_{\mathrm{B}} = -\sum_a \mathrm{i}(E_a)\nabla_{E_a} + \mathrm{i}(\kappa_{\mathrm{B}}),$$

where $\{E_a\}$ is a local orthonormal basic frame in Q and $\{\theta_a\}$ its g_Q-dual 1-form.

Proof. Fix $x \in M$ and choose an orthonormal basic frame $\{E_a\}$ so that $(\nabla E_a)_x = 0$ for all a. Since d_{B} is restriction of d, the first formula is trivial. Next we prove the second formula. Note that $\Omega^*_{\mathrm{B}}(\mathcal{F})$ is a transversal Clifford algebra with the Clifford multiplication defined as follows: if $\theta \in \Omega^1_{\mathrm{B}}(\mathcal{F})$ and $\phi \in \Omega^r_{\mathrm{B}}(\mathcal{F})$, then

$$\theta \cdot \phi = \theta \wedge \phi - \mathrm{i}(v)\phi, \tag{3.12}$$

where v is g_Q-dual vector of θ. Hence, if we use the properties (3.1), (3.2) and (3.12), then for any $\phi \in \Omega^r_{\mathrm{B}}(\mathcal{F})$ and $\psi \in \Omega^{r+1}_{\mathrm{B}}(\mathcal{F})$, we have that at x,

$$\begin{aligned}\langle d_{\mathrm{B}}\phi, \psi\rangle &= \sum_a \langle \theta_a \wedge \nabla_{E_a}\phi, \psi\rangle = \sum_a \langle E_a \cdot \nabla_{E_a}\phi, \psi\rangle \\ &= -\sum_a \langle \nabla_{E_a}\phi, E_a \cdot \psi\rangle = -\sum_a E_a\langle \phi, E_a \cdot \psi\rangle + \sum_a \langle \phi, E_a \cdot \nabla_{E_a}\psi\rangle \\ &= -\mathrm{div}_\nabla(v) + \sum_a \langle \phi, -\mathrm{i}(E_a)\nabla_{E_a}\psi\rangle,\end{aligned}$$

where $v \in \Gamma Q$ defined by the condition that $g_Q(v, w) = \langle \phi, w \cdot \psi\rangle$ for all $w \in \Gamma Q$. The first part of the last line in the above equation is proved as follows. At $x \in M$,

$$\mathrm{div}_\nabla(v) = \sum_a g_Q(\nabla_{E_a} v, E_a) = \sum_a E_a g_Q(v, E_a) = \sum_a E_a \langle \phi, E_a \cdot \psi\rangle.$$

By Green's theorem on the foliated Riemannian manifold [12], we get

$$\int_M \mathrm{div}_\nabla(v) = \langle\langle \kappa, v\rangle\rangle = \langle\langle \phi, \kappa \cdot \psi\rangle\rangle.$$

Hence, we have

$$\begin{aligned}\langle\langle d_{\mathrm{B}}\phi, \psi\rangle\rangle &= -\langle\langle \phi, \kappa_{\mathrm{B}} \cdot \psi\rangle\rangle + \langle\langle \phi, -\sum_a \mathrm{i}(E_a)\nabla_{E_a}\psi\rangle\rangle \\ &= \langle\langle \phi, -\sum_a \mathrm{i}(E_a)\nabla_{E_a}\psi + \mathrm{i}(\kappa_{\mathrm{B}})\psi\rangle\rangle,\end{aligned}$$

where κ_{B} is a basic component of κ. This finishes the proof. □

Note that the proof of Lemma 3.2 is different from that established in [1].

4. The first eigenvalue of $D_{\rm tr}$

Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a transversally oriented Riemannian foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$. Let $\mathrm{SO}(q) \to P \to M$ be the principal bundle of (oriented) transverse orthonormal framings. Then a *transverse spin structure* is a principal $\mathrm{Spin}(q)$-bundle $\tilde{P}$ together with two sheeted covering $\xi : \tilde{P} \to P$ such that $\xi(p \cdot g) = \xi(p)\xi_0(g)$ for all $p \in \tilde{P}$, $g \in \mathrm{Spin}(q)$, where $\xi_0 : \mathrm{Spin}(q) \to \mathrm{SO}(q)$ is a covering. In this case, the foliation $\mathcal{F}$ is called a *transverse spin foliation*. We then define the vector bundle S associated with $\tilde{P}$ by

$$S = \tilde{P} \times_{\mathrm{Spin}(q)} S_q, \tag{4.1}$$

where S_q is the irreducible spinor space associated to Q. The Hermitian metric on S is induced from g_Q, and the Riemannian connection ∇ on P defined by (2.1) can be lifted to one on $\tilde{P}$, in particular, to one on S, which will be denoted by the same letter. S is called the *foliated spinor bundle*. It is well known that the curvature transform R^S [9] is given as

$$R^S_{XY}\Phi = \frac{1}{4}\sum_{a,b} g_Q(R^\nabla_{XY}E_a, E_b)E_a \cdot E_b \cdot \Phi \quad \text{for } X, Y \in TM. \tag{4.2}$$

On the foliated spinor bundle S, we have

$$\mathcal{R}_\nabla = \tfrac{1}{4}\sigma^\nabla, \tag{4.3}$$

$$\sum_a E_a \cdot R^S_{XE_a}\Phi = -\frac{1}{2}\rho^\nabla(X) \cdot \Phi \tag{4.4}$$

for $X \in \Gamma Q$ [4]. From (3.9), we know that on an isoparametric transverse spin foliation with $\delta\kappa = 0$, the transverse Dirac operator $D_{\rm tr}$ satisfies

$$D_{\rm tr}^2 = \nabla_{\rm tr}^*\nabla_{\rm tr} + \tfrac{1}{4}\sigma^\nabla + \tfrac{1}{4}|\kappa|^2. \tag{4.5}$$

Now, we introduce a new connection $\overset{f}{\nabla}$ on S as

$$\overset{f}{\nabla}_X\Phi = \nabla_X\Phi + f\pi(X) \cdot \Phi \quad \text{for } X \in TM, \tag{4.6}$$

where f is a real-valued basic function on M and $\pi : TM \to Q$. Trivially, this connection $\overset{f}{\nabla}$ is a metric connection. Moreover, we have the following lemma.

Lemma 4.1. *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transversally oriented foliation $\mathcal{F}$ and bundle-like metric g_M with respect to $\mathcal{F}$. Then*

$$\langle\langle \overset{f}{\nabla}{}^*_{\rm tr}\overset{f}{\nabla}_{\rm tr}\Phi, \Psi\rangle\rangle = \langle\langle \overset{f}{\nabla}_{\rm tr}\Phi, \overset{f}{\nabla}_{\rm tr}\Psi\rangle\rangle$$

for all $\Phi, \Psi \in \Gamma S$.

Proof. Fix $x \in M$ and choose an orthonormal basic frame $\{E_a\}$ such that $(\nabla E_a)_x = 0$ for all a. Then, we have that at x,

$$\begin{aligned}\langle \overset{f}{\nabla}{}^*_{\rm tr}\overset{f}{\nabla}_{\rm tr}\Phi,\Psi\rangle &= -\sum_a \langle \overset{f}{\nabla}_{\rm tr}\overset{f}{\nabla}_{\rm tr}\Phi,\Psi\rangle + \langle \overset{f}{\nabla}_\kappa\Phi,\Psi\rangle\\ &= -\sum_a E_a\langle \overset{f}{\nabla}_{E_a}\Phi,\Psi\rangle + \sum_a\langle \overset{f}{\nabla}_{E_a}\Phi,\overset{f}{\nabla}_{E_a}\Psi\rangle + \langle\nabla_\kappa\Phi,\Psi\rangle + \langle f\kappa\cdot\Phi,\Psi\rangle\\ &= -\sum_a E_a\langle\nabla_{E_a}\Phi,\Psi\rangle - \sum_a\langle fE_a\cdot\Phi,\Psi\rangle + \sum_a\langle \overset{f}{\nabla}_{E_a}\Phi,\overset{f}{\nabla}_{E_a}\Psi\rangle\\ &\quad +\langle\nabla_\kappa\Phi,\Psi\rangle + \langle f\kappa\cdot\Phi,\Psi\rangle\\ &= -{\rm div}_\nabla(V) - {\rm div}_\nabla(fW) + \sum_a\langle \overset{f}{\nabla}_{E_a}\Phi,\overset{f}{\nabla}_{E_a}\Psi\rangle + \langle\nabla_\kappa\Phi,\Psi\rangle\\ &\quad +\langle f\kappa\cdot\Phi,\Psi\rangle,\end{aligned}$$

where $V, W \in \Gamma Q\otimes C$ are defined by the conditions that $g_Q(V,Z) = \langle\nabla_Z\Phi,\Psi\rangle$ and $g_Q(W,Z) = \langle Z\cdot\Phi,\Psi\rangle$ for all $Z\in\Gamma Q$. The last line is proved as follows: At $x\in M$,

$${\rm div}_\nabla(V) = \sum_a g_Q(\nabla_{E_a}V, E_a) = \sum_a E_a g_Q(V,E_a) = \sum_a E_a\langle\nabla_{E_a}\Phi,\Psi\rangle.$$

Similarly, we have ${\rm div}_\nabla(fW) = \sum_a E_a\langle fE_a\cdot\Phi,\Psi\rangle$.

By Green's theorem on the foliated Riemannian manifold [12]

$$\int_M {\rm div}_\nabla(V) = \langle\langle\kappa, V\rangle\rangle = \langle\langle\nabla_\kappa\Phi,\Psi\rangle\rangle. \tag{4.7}$$

Similarly, we have $\int_M {\rm div}_\nabla(fW) = \langle\langle f\kappa\cdot\Phi,\Psi\rangle\rangle$. By integrating, we obtain our result. □

On the other hand, by using (3.6) and (4.6), we have

$$\begin{aligned}\overset{f}{\nabla}{}^*_{\rm tr}\overset{f}{\nabla}_{\rm tr}\Phi &= -\sum_a \overset{f}{\nabla}_{E_a}\overset{f}{\nabla}_{E_a}\Phi + \overset{f}{\nabla}_\kappa\Phi = -\sum_a\nabla_{E_a}\nabla_{E_a}\Phi + \nabla_\kappa\Phi - f\sum_a E_a\cdot\nabla_{E_a}\Phi\\ &\quad -\sum_a\nabla_{E_a}(fE_a\cdot\Phi) + f\kappa\cdot\Phi - f^2\sum_a E_a\cdot E_a\cdot\Phi.\end{aligned}$$

From the definition of Clifford multiplication and (3.2), we have

$$\begin{aligned}\overset{f}{\nabla}{}^*_{\rm tr}\overset{f}{\nabla}_{\rm tr}\Phi &= -\sum_a\nabla_{E_a}\nabla_{E_a}\Phi + \nabla_\kappa\Phi - 2f\left(\sum_a E_a\cdot\nabla_{E_a}\Phi - \frac{1}{2}\kappa\cdot\Phi\right)\\ &\quad -\sum_a E_a(f)E_a\cdot\Phi + qf^2\Phi.\end{aligned}$$

From this equation, we have

$$\overset{f}{\nabla}{}^*_{\rm tr}\overset{f}{\nabla}_{\rm tr}\Phi = \nabla^*_{\rm tr}\nabla_{\rm tr}\Phi - 2fD_{\rm tr}\Phi - {\rm grad}_\nabla(f)\cdot\Phi + qf^2\Phi, \tag{4.8}$$

where ${\rm grad}_\nabla(f) = \sum_a E_a(f)E_a$ is a transversal gradient of f. From (4.5) and (4.8), we have

$$\overset{f}{\nabla}{}^*_{\rm tr}\overset{f}{\nabla}_{\rm tr}\Phi = D^2_{\rm tr}\Phi - 2fD_{\rm tr}\Phi - {\rm grad}_\nabla(f)\cdot\Phi + (qf^2 - \tfrac{1}{4}(\sigma^\nabla + |\kappa|^2))\Phi. \tag{4.9}$$

Let $D_{\rm tr}\Phi = \lambda\Phi(\Phi \neq 0)$. From (4.9) and Lemma 4.1, we have

$$\|\overset{f}{\nabla}_{\rm tr}\Phi\|^2 = \int_M \left(\lambda^2 - 2f\lambda + qf^2 - \frac{1}{4}(\sigma^\nabla + |\kappa|^2)\right)|\Phi|^2 - \int_M \langle {\rm grad}_\nabla(f)\cdot\Phi, \Phi\rangle.$$

Note that for all $X \in \Gamma Q$ and $\Phi \in \Gamma S$,

$$\langle X\cdot\Phi, \Phi\rangle = \overline{\langle\Phi, X\cdot\Phi\rangle} = -\overline{\langle X\cdot\Phi, \Phi\rangle}. \tag{4.10}$$

Hence, (4.10) implies that $\langle {\rm grad}_\nabla(f)\cdot\Phi, \Phi\rangle$ is a pure imaginary. Hence, we have

$$\|\overset{f}{\nabla}_{\rm tr}\Phi\|^2 = \int_M \left(\lambda^2 - 2f\lambda + qf^2 - \frac{1}{4}(\sigma^\nabla + |\kappa|^2)\right)|\Phi|^2, \tag{4.11}$$

$$\langle {\rm grad}_\nabla(f)\cdot\Phi, \Phi\rangle = 0. \tag{4.12}$$

If we put $f = \lambda/q$, then from (4.11), we have

$$\|\overset{f}{\nabla}_{\rm tr}\Phi\|^2 = \int_M \left(\frac{q-1}{q}\lambda^2 - \frac{1}{4}K_\sigma\right)|\Phi|^2, \tag{4.13}$$

where $K_\sigma = \sigma^\nabla + |\kappa|^2$. From (4.13), we have the following theorem.

Theorem 4.2. *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with an isoparametric transverse spin foliation $\mathcal{F}$ of codimension $q > 1$ and bundle-like metric g_M with respect to $\mathcal{F}$. Assume that the mean curvature κ of $\mathcal{F}$ satisfies $\delta\kappa = 0$ and $K_\sigma \geq 0$. Then the eigenvalue λ of the transverse Dirac operator $D_{\rm tr}$ satisfies*

$$\lambda^2 \geq \frac{1}{4}\frac{q}{q-1}K_\sigma^0,$$

where $K_\sigma^0 = \min K_\sigma$.

Remark. If $\mathcal{F}$ is a point foliation, then the transversal Dirac operator is just a Dirac operator on an ordinary manifold. Therefore, Theorem 4.2 is a generalization of the result on an ordinary manifold (cf. [4]).

Corollary 4.3. *In addition to assumptions in Theorem* 4.2, *if the transverse scalar curvature is zero, then we get*

$$\lambda^2 \geq \frac{q}{4(q-1)}|\kappa|_0^2,$$

where $|\kappa|_0 = \min|\kappa|$.

5. The limiting case

In this section, we study the foliated Riemannian manifold which admits a non-zero transversal spinor Ψ_1 such that $D_{\rm tr}\Psi_1 = \lambda_1\Psi_1$ with $\lambda_1^2 = \frac{1}{4}(q/(q-1))K_\sigma^0$. We define

$\mathrm{Ric}^{f}_{\nabla} : \Gamma Q \otimes S \to S$ by

$$\mathrm{Ric}^{f}_{\nabla}(X \otimes \Psi) = \sum_a E_a \cdot R^{f}_{XE_a}\Phi, \tag{5.1}$$

where R^f is the curvature tensor with respect to $\overset{f}{\nabla}$. By long calculation, for $X \in \Gamma Q$ and $\Phi \in \Gamma S$ we have

$$\sum_a E_a \cdot R^{f}_{XE_a}\Phi = \sum_a E_a \cdot R^{S}_{XE_a}\Phi - qX(f)\Phi \\ -\mathrm{grad}_{\nabla}(f) \cdot X \cdot \Phi + 2(q-1)f^2 X \cdot \Phi. \tag{5.2}$$

From (4.4) and (5.2), we have

$$\mathrm{Ric}^{f}_{\nabla}(X \otimes \Phi) = -\tfrac{1}{2}\rho^{\nabla}(X) \cdot \Phi + 2(q-1)f^2 X \cdot \Phi - qX(f)\Phi - \mathrm{grad}_{\nabla}(f) \cdot X \cdot \Phi \tag{5.3}$$

for $X \in \Gamma Q$. From (5.3), we have the following proposition.

Proposition 5.1. *If M admits a non-zero transversal Killing spinor* Φ (*i.e., it is defined by* $\nabla^{f}{}_{\mathrm{tr}}\Phi = 0$), *then f is constant.*

Proof. If $\overset{f}{\nabla}_X\Phi = 0$ for any $X \in \Gamma Q$, then $\mathrm{Ric}^{f}_{\nabla} = 0$. Hence, from (5.3), we have

$$-\tfrac{1}{2}\rho^{\nabla}(X) \cdot \Phi + 2(q-1)f^2 X \cdot \Phi - \mathrm{grad}_{\nabla}(f) \cdot X \cdot \Phi - qX(f)\Phi = 0. \tag{5.4}$$

If we put $X = \mathrm{grad}_{\nabla}(f)$, then we have

$$\langle(-\tfrac{1}{2}\rho^{\nabla}(X) + 2(q-1)f^2\,\mathrm{grad}_{\nabla}(f)) \cdot \Phi, \Phi\rangle = (q-1)|\mathrm{grad}_{\nabla}(f)|^2|\Phi|^2. \tag{5.5}$$

From (4.10), the left-hand side is pure imaginary, but the right-hand side is real. Therefore, both sides are zero. Hence, if $q \geq 2$, then we have

$$\mathrm{grad}_{\nabla}(f) = 0.$$

That is, $X(f) = 0$ for any $X \in \Gamma Q$. Since f is a basic function (i.e., $X(f) = 0$ for any $X \in \Gamma L$), f is constant. □

We now consider the limiting case. Let $\lambda_1^2 = \frac{1}{4}(q/(q-1))K^0_{\sigma}$. From (4.13), we have

$$\|\overset{f_1}{\nabla}_{\mathrm{tr}}\Psi_1\|^2 = \int_M \frac{1}{4}(K^0_{\sigma} - K_{\sigma})|\Psi_1|^2, \tag{5.6}$$

where $f_1 = \lambda_1/q$. From this equation, we have

$$K_{\sigma} = K^0_{\sigma}, \qquad \overset{f_1}{\nabla}_{\mathrm{tr}}\Psi_1 = 0. \tag{5.7}$$

From (5.3), we have

$$\rho^{\nabla}(X) = 4(q-1)f_1^2 X = \frac{1}{q}K^0_{\sigma}X \quad \text{for } X \in \Gamma Q. \tag{5.8}$$

This means that $\mathcal{F}$ is a transversally Einsteinian with constant transversal scalar curvature σ^{∇}. Hence, we have the following theorem.

Theorem 5.2. *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with an isoparametric transverse spin foliation $\mathcal{F}$ of codimension $q > 1$ and a bundle-like metric g_M with respect to $\mathcal{F}$. Assume that the mean curvature κ satisfies $\delta\kappa = 0$ and $K_\sigma > 0$. If there exists an eigenspinor field Ψ_1 of transversal Dirac operator D_{tr} for the eigenvalue $\lambda_1^2 = (q/4(q-1))K_\sigma^0$, then $\mathcal{F}$ is a minimal, transversally Einsteinian with constant transversal scalar curvature.*

Proof. If we compare (2.2) with (5.8), we know that $|\kappa| = 0$. This implies that $\mathcal{F}$ is minimal. □

Remark. Theorem 5.2 implies that the estimate in Theorem 4.2 is not sharp. In fact, if the foliation $\mathcal{F}$ is not minimal, then $\lambda^2 > (q/4(q-1))K_\sigma^0$. So, we can assume that a sharper estimate than the one in Theorem 4.2 exists.

From now on, if we let $\mathcal{F}$ be a minimal foliation, then on $\Omega_{\mathrm{B}}^*(\mathcal{F})$, we have

$$D_{\mathrm{b}} = d_{\mathrm{B}} + \delta_{\mathrm{B}}, \qquad \Delta_{\mathrm{B}} = D_{\mathrm{b}}^2. \tag{5.9}$$

Hence, the eigenvalue μ of Δ_{B} satisfies the inequality in Theorem 4.2. Moreover, for any basic r-form $\phi \in \Omega_{\mathrm{B}}^r(\mathcal{F})$, we have, by long calculation,

$$\sum_a E_a \cdot \phi \cdot E_a = (-1)^{r-1}(q-2r)\phi, \tag{5.10}$$

where $\{E_a\}$ is an orthonormal basic frame field of Q. From (5.7), (5.9) and (5.10), we have

$$D_{\mathrm{b}}(\phi \cdot \Psi_1) = (d_{\mathrm{B}} + \delta_{\mathrm{B}})\phi \cdot \Psi_1 + (-1)^r (q-2r) f \phi \cdot \Psi_1 = (d_{\mathrm{B}} + \delta_{\mathrm{B}})\phi \cdot \Psi_1 + (-1)^r (q-2r)\frac{\lambda_1}{q}\phi \cdot \Psi_1,$$

where Ψ_1 is the non-zero eigenspinor corresponding to λ_1. If ϕ is a basic harmonic form, then we have

$$D_{\mathrm{b}}(\phi \cdot \Psi_1) = (-1)^r \left(1 - \frac{2r}{q}\right)\lambda_1 \phi \cdot \Psi_1. \tag{5.11}$$

This leads us to the following proposition (cf. [6]):

Proposition 5.3. *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with an isoparametric transverse spin foliation $\mathcal{F}$ with codimension $q > 1$ and a bundle-like metric g_M with respect to $\mathcal{F}$. Assume that the mean curvature κ satisfies $\delta\kappa = 0$ and $K_\sigma > 0$. If M admits an eigenspinor field Ψ_1 associated with λ_1 such that $\lambda_1^2 = (q/4(q-1))K_\sigma^0$, then any basic harmonic form kills the eigenspinor Ψ_1.*

Proof. By Theorem 5.2, $\mathcal{F}$ is minimal. Hence, Eq. (5.11) implies that $(-1)^r(1-(2r/q))\lambda_1$ is an eigenvalue of D_{b}. But since λ_1 is the first eigenvalue of D_{b}, this is impossible. So, we have $\phi \cdot \Psi_1 = 0$. □

Corollary 5.4. *Under the same assumption as in Proposition* 5.3, *we have*

$$H^1_{\mathrm{B}}(\mathcal{F}) = 0,$$

where $H^r_{\mathrm{B}}(\mathcal{F})$ *is the basic cohomology*.

Proof. For a basic harmonic 1-form ϕ, we have $\phi\cdot\Psi_1 = 0$. Hence, $0 = \phi\cdot\phi\cdot\Psi_1 = -|\phi|^2\Psi_1$. Since $\Psi_1 \neq 0$, we have $\phi = 0$. □

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